Chapter 1

Sets[1]

1.1 Writing sets down

Sets

In mathematics, any collection of mathematical objects will be called a **set**. For example,

$$\{1, 2, 3\},$$

denotes the set that contains the numbers $1, 2$ and 3 (and nothing else). As well as numbers, all sort of things, such as matrices, vectors, functions and even sets themselves, count as mathematical objects.

As to what really constitutes a 'mathematical object' and thus what is really meant by a set, we could define these things precisely, but to do so here would lead us down dark alleyways.

Basically, you should trust that whenever your lecturer writes down a set, he or she is providing you with a genuine set and not some counterfeit knockoff.

Typically, sets are denoted using capital letters, such as A, B, C, or X, Y, Z, etc. The members of a set are called its **elements**. If A denotes the set above, then $1, 2$ and 3 are elements of A (and nothing else). If X is a set and x is an element of X, then we write

$$x \in X.$$

The symbol $\in$ should be interpreted as 'is an element of'. If x and y are both elements of X, then we can write

$$x, y \in X,$$

to denote this, and likewise for any number of elements. If x is not an element of X, then we put a line through the $\in$ symbol and write $x \notin X$.

[1] This chapter is part of a university mathematics guide that is still in development. It is intended principally for first years, but also as a reference for people in later years. It has been approved for use by the UCD Maths Support Centre.

Example 1.1. Let $A = \{1, 2, 3\}$. Then $1, 2, 3 \in A$, but $4 \notin A$.

Of course, there are lots of other things that are not in A. For example, $5, 6, 7, 8 \notin A$, and so on.

Defining sets using lists and dots

A set can be defined by listing its elements within curly brackets. We have already seen an example of this above. If we wanted to write down the set of all integers between 1 and 15 (inclusive), then we could do so by writing

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\},$$

however, if the context is clear, we often spare ourselves the trouble of writing out every number by using an **ellipsis**, a series of (usually) three dots ($\ldots$). Thus the set above could also be denoted by

$$\{1, 2, 3, \ldots, 15\},$$

Regrettably, 'ellipses' is the plural form of two words meaning different things, namely 'ellipsis' and 'ellipse'.

and everyone would understand the meaning. Ellipses become particularly useful when it would become truly nightmarish or even impossible to define a set by literally listing every element of it, such as the set

$$\{1, 2, 3, \ldots, 1000\},$$

of all integers between 1 and 1000, or the set

$$\{1, 2, 3, \ldots, n\},$$

If $n = 1$ or $n = 2$, then we interpret

$$\{1, 2, 3, \ldots, n\},$$

as $\{1\}$ or $\{1, 2\}$, respectively.

of all integers between 1 and n, where n is some indeterminate (it has no specific value) integer greater than or equal to 1. In the last example, n could be 7, in which case explicitly listing all elements of the set is quite easy, but equally n could be 10^{90}, in which case explicitly listing all the elements would be tricky because 10^{90} exceeds current estimates of the number of atoms in the observable universe.

Repetitions and order do not matter (too much)

If a set is defined by a list, and one or more of the elements are repeated, then these repetitions are ignored. Moreover, the order in which the elements are listed does not matter, though for clarity, if there is a natural order to the elements then it is helpful to use it. Thus the set

$$\{1, 7, 2, 4, 1, 7, 3, 2, -5\},$$

defined using a motley dishevelled jumble of numbers, is equal to the much clearer and tidier looking

$$\{-5, 1, 2, 3, 4, 7\}.$$

In particular, given two distinct mathematical objects a and b (e.g. they could be two distinct numbers), the set $\{a, b\}$ is called an **unordered pair**. It is called unordered precisely because (appeals to clarity aside) the order in which the elements are listed does not matter:

$$\{a, b\} = \{b, a\}.$$

Some examples of sets in mathematics

In mathematics one frequently encounters sets containing infinitely many elements. Ellipses (the dots, not the geometric figures) can be used to denote them sometimes.

Example 1.2.

1. The **set of natural numbers** is denoted by

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

2. The **set of integers** is denoted by

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

At the other end of the scale, we have the smallest set of all.

Example 1.3. The set having no elements whatsoever is called the **empty set**, and is denoted by $\{\}$ or $\varnothing$.

It may seem peculiar to consider such a thing, but it isn't really. After all, when we are counting things, the number 0 can be used in situations where there is nothing to count in the first place, and these days nobody minds the number 0. One may not think so at first glance, but the empty set can be quite useful sometimes. Just to be clear, the empty set is not nothing. Instead, it is the set that contains nothing. They're different!

Though throughout the ages, the seemingly humble 0 has had a far more swashbucking career than you might imagine.

Defining sets using rules

Let's move on. We have seen how sets can be defined by listing the elements inside or by using ellipses. Quite often though this notation is inadequate to accurately describe many sets. Instead, we use rules to determine the elements that belong to these sets.

This is often done by taking an existing set, say $\mathbb{N}$, and specifying a new set by demanding that its elements belong to the set we started with **and** obey an additional rule. For example,

$$\{n \in \mathbb{N} \ : \ 12 \leqslant n \leqslant 17\}\,,$$

denotes the set of all $n \in \mathbb{N}$ satisfying an additional rule, namely that n is both greater than or equal to 12 and less than or equal to 17. In other words, we have just defined the set

$$\{12, 13, 14, 15, 16, 17\}.$$

The colon above means 'such that', so you should interpret the notation above as

'the set of all $n \in \mathbb{N}$ such that $12 \leqslant n \leqslant 17$'.

Sometimes people use a vertical line | instead of a colon to denote 'such that'. Also, you may find variations on the notation above, such as

$$\{n \ : \ n \in \mathbb{N} \text{ and } 12 \leqslant n \leqslant 17\}\,,$$

which should be read as

'the set of all n such that $n \in \mathbb{N}$ and $12 \leqslant n \leqslant 17$'.

The meaning of this is exactly the same as above, and we get exactly the same set! Sometimes different notation produces the same thing. Lecturers use slightly different notation sometimes because each one has his or her own individual style, but the differences should still remain within accepted convention. In any case, if you are in any doubt at all about notation, ask for clarification!

As we saw, the elements of the set above could be listed explicitly quite easily, but in other situations this is difficult or impossible, and in yet others it may simply be clearer or more convenient to use a rule.

Example 1.4.

1. The set
$$\{p \in \mathbb{N} \ : \ p \text{ is a prime number}\},$$
denotes the set of all prime numbers $2, 3, 5, 7, 11, 13, 17, 19, \ldots$.

2. The set
$$\{n \in \mathbb{Z} \ : \ n \text{ is an odd number}\},$$
denotes the set of all odd integers $1, -1, 3, -3, 5, -5, \ldots$.

Sometimes we allow ourselves more complicated expressions on the left hand side of the colon. This can help when the prospect of listing elements looks very unattractive, as in the next example.

The elements of $\mathbb{Q}$ can be listed explicitly, but not in any way that is easy on the eye.

Example 1.5. The set

$$\mathbb{Q} = \left\{\frac{m}{n} \ : \ m, n \in \mathbb{Z} \text{ and } n \neq 0\right\},$$

is the set of **rational numbers**, that is, the set of all numbers that can be written as the quotient of two integers m and n, where n is non-zero.

Thus $\frac{1}{2}, \frac{11}{4}, -\frac{79}{3} \in \mathbb{Q}$ etc., but famously, in a result that goes back more than two thousand years to the ancient Greeks, we know that $\sqrt{2} \notin \mathbb{Q}$, in other words, the (positive) square root of 2 is not a rational number.

There are other well known numbers that are not rational numbers and thus do not belong to $\mathbb{Q}$. For example $\pi \notin \mathbb{Q}$ and $\mathrm{e} \notin \mathbb{Q}$, where e is the base of the natural logarithm. Despite not being rational, all these numbers are examples of what we call **real numbers**.

Example 1.6. The set of real numbers is denoted by $\mathbb{R}$.

Properly defining the set of real numbers takes some work, and we will don't do so here. Very roughly speaking, the set of real numbers is a mathematical model of 1-dimensional physical reality. It contains all the numbers that you could possibly need for doing finance or for measuring quantities such as length, area, volume, mass, speed, and so on.

Quite often in calculus and mathematical analysis courses, it is necessary to work with so-called **intervals** of $\mathbb{R}$.

Example 1.7. Let $x, y \in \mathbb{R}$, with $x \leqslant y$.

1. The set
$$(x, y) = \{t \in \mathbb{R} : x < t < y\},$$
is known as the **open interval** having endpoints x and y. It is the set of all real numbers t that lie **strictly** between x and y. Thus $\frac{3}{2} \in (1, 2)$, because $1 < \frac{3}{2} < 2$, but $5 \notin (1, 2)$ because it is not true that $1 < 5 < 2$.

2. The set
$$[x, y] = \{t \in \mathbb{R} : x \leqslant t \leqslant y\},$$
is known as the **closed interval** having endpoints x and y. It is the set of all real numbers t that lie between x and y, but not strictly so.

The difference between (x, y) and $[x, y]$ is that the former does not include the end points, while the latter does: $x, y \notin (x, y)$ and $x, y \in [x, y]$. For example, $3, 7 \notin (3, 7)$, but $3, 7 \in [3, 7]$.

This may not seem like a terribly big deal at first glance, but this difference has profound consequences. Correctly distinguishing between the two comes highly recommended. Notice that if $x = y$ then $(x, y) = \varnothing$ and $[x, y] = \{x\}$. Indeed, observe that there are **no** real numbers t such that $x < t < x$. Thus the set of all such numbers must be empty. There is only one real number t such that $x \leqslant t \leqslant x$, namely x itself.

Every real number x has the property that x^2 is non-negative. The 'imaginary' number i, which was introduced long ago, has the property that $\mathrm{i}^2 = -1$. Despite being 'imaginary' and not 'real', this number has an enormous range of applications to problems that are perfectly real. The set of **complex numbers** gives us another opportunity to define a set using an expression on the left hand side of the colon.

Example 1.8. The set
$$\mathbb{C} = \{x + \mathrm{i}y : x, y \in \mathbb{R}\}.$$

is the set of **complex numbers**.

When is a rule a rule?

As we have seen, sets can be defined using rules. But what constitutes a valid rule? One could become overly enthusiastic and try to define the set

$$\{n \in \mathbb{N} \;:\; n \text{ is feeling lonely today}\}\,.$$

However, this endeavour begins and ends in wretched failure. As this example shows, clearly not everything masquerading as a rule will be genuine.

It turns out that there is a series of other rules that can determine when a rule is a proper rule or not, but spelling this process out explicitly requires some reasonably advanced mathematics in its own right, and doing so here would be counterproductive.

Instead, it is better to start off by trusting your lecturers to define the sets they need using valid rules and, over time, to use these examples to build up your own sense of what is valid and what isn't. Roughly speaking, a rule will be valid if, for every element under consideration, it is absolutely clear whether the element satisfies the rule or not. For example, given $p \in \mathbb{N}$, either p is a prime number or it is not – there is no middle ground or opportunity for ambiguity here.

Restoring order

Recall our **unordered pairs**,

$$\{a, b\} \;=\; \{b, a\}.$$

There is also the notion of an ordered pair. An **ordered pair** is a pair of mathematical objects a and b, and is denoted

$$(a, b).$$

It is regrettable that the notation for ordered pairs and open intervals of real numbers looks the same.

This means that there is scope for ambiguity, but usually it is possible to tell one from the other from the context.

The distinction between unordered pairs and ordered pairs is that if a and b are distinct, then

$$(a, b) \;\neq\; (b, a),$$

in other words, the order in which a and b appear in the pair matters.

Ordered pairs are used, for example, to represent points in the 2-dimensional cartesian plane. The ordered pair $(1, 3)$ can be used to represent the point having x-coordinate 1 and y-coordinate 3. Of course, this point is different from the point having x-coordinate 3 and y-coordinate 1, and so if we are represent such points using pairs of numbers (and doing so is incredibly useful), then the order in which the numbers appear in the pair has to matter.

Ordered pairs also count as mathematical objects, so we can form sets of them.

Example 1.9. The set

$$\mathbb{R}^2 = \{(a, b) : a, b \in \mathbb{R}\},$$

is the set of all ordered pairs of real numbers. This set can be used to represent the entire 2-dimensional cartesian plane.

Ordered pairs are defined in such a way to ensure that $(a, b) \neq (b, a)$ whenever a and b are different. For the purposes of these notes however, the definition (while relatively simple) has been conveniently tied up in the attic where nobody can see it.

Sets of sets, and the joys of meaning things literally

Remember that sets themselves count as mathematical objects. This means that sets can be elements of other sets.

Example 1.10. The set

$$\{0, \{1, 2\}, \{5, 7, 10, 11\}\},$$

is a set that contains precisely **three** objects, namely the number 0 and the sets $\{1, 2\}$ and $\{5, 7, 10, 11\}$. It is **not** the same as

$$\{\{0, 1, 2\}, 5, 7, 10, 11\},$$

which is a set containing **five** objects, namely the set $\{0, 1, 2\}$ and the numbers $5, 7, 10$ and 11.

As you can see from the example above, the placement of curly brackets matters! It is unlikely that you will encounter sets containing other sets in the first one or two years of a university mathematics degree, but you will see them a lot in subsequent years.

It is worth stressing the importance of placing brackets correctly and of being precise in one's mathematical notation in general. In everyday language it is common for us to not literally mean what we say. For example, when someone says 'see you in a second', they probably don't literally mean that. In mathematics however it is important to literally mean what we say or write, so that ambiguity (which is sometimes necessary in life but is poison for mathematics) can be avoided.

We are familiar with the use of brackets in arithmetic to determine the order in which calculations are made, for example

$$(1 + 3) \times 5 \qquad \text{and} \qquad 1 + (3 \times 5)$$

mean different things: the first yields $4 \times 5 = 20$ and the second $1 + 15 = 16$. In the same way, when dealing with sets, putting brackets in different places yield different sets, and just like arithmetic, it is important to always keep this in mind.

Example 1.11. The set $\{\varnothing\}$ is a set containing precisely one element, namely the empty set $\varnothing$. It is **not** the same as $\varnothing$, because this second set contains no elements at all!

Of course, since $\{\varnothing\}$ is a set, we can form the set $\{\{\varnothing\}\}$, that is, the set that contains the set that contains the empty set. We could go on, but let's not.

Actually there is a time and a place for this sort of carry on, namely a university level course in so-called 'set theory'.

I solemnly promise that the apparently useless exercise of making sets out of sets out of sets, and so on, reaps huge dividends for mathematics.

1.2 Doing things with sets

Now that we know how to define sets, we come to the business of doing something with them.

Subsets, supersets and equality

We begin by looking at how to compare one set with another. Making comparisons is important in life (I have twice as much money as you), and without making comparisons one could not do a lot of mathematics.

We are familiar with the idea of comparing two numbers and finding that one is less than or equal to the other, for example, $1 \leqslant 3$ or $-5 \leqslant 20$. There is an analogous way of comparing one set with another. Let A and B be two sets. We say that A is a **subset** of B, written

$$A \subseteq B,$$

if every element of A is also an element B.

Given two numbers a and b, if a is less than or equal to b, written $a \leqslant b$, then equally we can say that b is greater than or equal to a, or $b \geqslant a$. When it comes to sets, if A is a subset of B, then equally we can say that B is a **superset** of A, and write

$$B \supseteq A.$$

Example 1.12.

1. The set $\{-9, 58\}$ has two elements, namely -9 and 58. Both these numbers are also elements of $\{-11, -9, 1, 14, 27, 58\}$. Thus,
$$\{-9, 58\} \subseteq \{-11, -9, 1, 14, 27, 58\}.$$
Equally, $\{-11, -9, 1, 14, 27, 58\} \supseteq \{-9, 58\}$.
2. Every natural number is also an integer, so we have $\mathbb{N} \subseteq \mathbb{Z}$.
3. Every integer $m \in \mathbb{Z}$ is also a rational number, because
$$m = \frac{m}{1},$$
and $1 \in \mathbb{Z}$ is not zero. Therefore, $\mathbb{Z} \subseteq \mathbb{Q}$.
4. Every rational number is also a real number, so $\mathbb{Q} \subseteq \mathbb{R}$.
5. Every real number $x \in \mathbb{R}$ is also a complex number, because
$$x = x + \mathrm{i}0,$$
and $0 \in \mathbb{R}$. Therefore, $\mathbb{R} \subseteq \mathbb{C}$.

If we have a series of sets, each one a subset of the next, then we can string these comparisons together to form a chain. For instance,

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

What about equality? As alluded to above, but never formally stated, we say that two sets A and B are **equal**, written rather unsurprisingly as $A = B$, when they contain precisely the same elements. In other words,

$A = B$ when every element of A is an element of B, and every element of B is also an element of A. If A and B are not equal then we write $A \neq B$.

Notice that if $A = B$ then we also have $A \subseteq B$ (and $B \subseteq A$), because in this situation, every element of A is also an element of B. Conversely, if for sets A and B, we have $A \subseteq B$ and $B \subseteq A$, then $A = B$. Compare this with what happens with numbers: if a and b are real numbers and $a \leqslant b$ and $b \leqslant a$, then $a = b$.

Example 1.13. In Example 1.12, none of the sets under consideration are equal.

1. The set $\{-11, -9, 1, 14, 27, 58\}$ has an element (in fact several) that is not in $\{-9, 58\}$. For example,

   $$-11 \in \{-11, -9, 1, 14, 27, 58\} \qquad \text{but} \qquad -11 \notin \{-9, 58\}.$$

   Therefore,

   $$\{-9, 58\} \neq \{-11, -9, 1, 14, 27, 58\}.$$

2. The set of integers $\mathbb{Z}$ has an element (in fact, infinitely many), for example -1, that is not an element of $\mathbb{N}$. Thus $\mathbb{N} \neq \mathbb{Z}$.
3. Since, for example, $\frac{1}{2} \in \mathbb{Q}$ but $\frac{1}{2} \notin \mathbb{Z}$, we have $\mathbb{Z} \neq \mathbb{Q}$.
4. Because, for instance, $\sqrt{2} \in \mathbb{R}$ but $\sqrt{2} \notin \mathbb{Q}$, we have $\mathbb{Q} \neq \mathbb{R}$.
5. Finally, $\mathrm{i} \in \mathbb{C}$ but $\mathrm{i} \notin \mathbb{R}$, so $\mathbb{R} \neq \mathbb{C}$.

One last word on subsets. We know that A is a subset of B if every element of A is also an element B. If this is **not** the case, then there must be some element of A that is not an element of B. In this instance we say, unsurprisingly, that A is not a subset of B, and write $A \not\subseteq B$. Equally, we can write $B \not\supseteq A$. From the examples above, we have

$$\{-11, -9, 1, 14, 27, 58\} \not\subseteq \{-9, 58\}, \quad \mathbb{Z} \not\subseteq \mathbb{N}, \quad \mathbb{Q} \not\subseteq \mathbb{Z},$$

and so on.

But there is an important difference between comparing sets and comparing numbers. Given two real numbers a and b, if a is **not** less than or equal to b, written $a \nleqslant b$, then we conclude that b must be less than a.

We cannot draw the analogous conclusion when comparing sets: if $A \nsubseteq B$, then we cannot conclude that $B \subseteq A$.

Example 1.14. Take the sets $\{-9, 58\}$ and $\{-9, 27\}$. Since $58 \in \{-9, 58\}$ but $58 \notin \{-9, 27\}$, we have

$$\{-9, 58\} \ \nsubseteq \ \{-9, 27\},$$

and as $27 \in \{-9, 27\}$ but $27 \notin \{-9, 58\}$, we have

$$\{-9, 27\} \ \nsubseteq \ \{-9, 58\},$$

as well.

If $A \nsubseteq B$ and $B \nsubseteq A$ then we call A and B **incomparable**. Like apples and oranges in everyday life, or one Jedward twin and the other one, sometimes things in mathematics simply cannot be compared.

Unions, intersections and complements

Imagine you have a list of email addresses. Some of them belong to people who like coffee, others doughnuts (and some both). For whatever reason, you may want email those people who like coffee or doughnuts (or both) about some exciting new product that somehow incorporates either a coffee or a doughnut. Instead, if your product somehow incorporates both coffee and doughnuts in some unholy mixture, then you may want to email only those people who like both coffee and doughnuts. Finally, if you have a product that includes coffee but not doughnuts (for example, coffee) you may decide that those who like coffee but in fact **dislike** doughnuts are the only people exclusive enough to receive your exciting news.

Mathematics has the necessary tools to construct the respective email lists. Let A and B be sets. We define the **union** of A and B to be the set of all elements that are either in A or in B:

$$A \cup B \ = \ \{x \ : \ x \in A \text{ or } x \in B\}\,.$$

The union is the set of all elements that are in **at least** one of A or B: elements are allowed to be in both as well.

Example 1.15.

1. $\{6,7\} \cup \{7,12,13\} = \{6,7,12,13\}$.
2. Let $[5,10]$ and $[8,16]$ be closed intervals of $\mathbb{R}$. Then their union is another closed interval:
$$[5,10] \cup [8,16] \; = \; [5,16].$$

Alongside the union, we define the **intersection** of A and B to the set of all elements that are in both A **and** B:

$$A \cap B \; := \; \{x \; : \; x \in A \text{ and } x \in B\}.$$

Example 1.16. Let's consider again the sets from Example 1.15.

1. $\{6,7\} \cap \{7,12,13\} = \{7\}$.
2. $[5,10] \cap [8,16] \; = \; [8,10]$.

Two sets A and B are generally under no obligation to share any elements at all. When this happens, and it often does, then the intersection of the two is equal to the empty set. For example,

$$\{1,2\} \cap \{3,4\} \; = \; \varnothing.$$

There are **no** elements that are in both $\{1,2\}$ **and** $\{3,4\}$, hence the set of all such (non-existent) elements is empty.

Finally, we define the **complement** of B with respect to A, or equivalently the **difference** of A and B, to be the set of all elements that are in A but **not** in B:

$$A \setminus B \; = \; \{x \; : \; x \in A \text{ and } x \notin B\}.$$

Example 1.17. Let's return to the sets from Example 1.15 for a final time.

1. $\{6,7\} \setminus \{7,12,13\} = \{6\}$ and $\{7,12,13\} \setminus \{6,7\} = \{12,13\}$.

2. We have

$$[5,10]\setminus[8,16] \;=\; \{t\in\mathbb{R} \;:\; 5\leqslant t<8\}\,,$$

and

$$[8,16]\setminus[5,10] \;=\; \{t\in\mathbb{R} \;:\; 10<t\leqslant 16\}\,.$$

We end with a small remark that happens to be pretty big. Remember that two sets are said to be equal if they contain exactly the same elements. In Examples 1.15 to 1.17, a series of equalties between sets are given, for instance, in Example 1.15 (1) it is stated that

$$\{6,7\}\cup\{7,12,13\}=\{6,7,12,13\}.$$

Each such statement is a claim that needs to be justified. Mathematics is very strict about this: don't make a mathematical claim unless you can back it up with hard proof! To justify these equalities, we need to make sure that every element in a given set on left hand side of the equality sign is an element of the corresponding set on the right hand side, and vice-versa. In part (1) of Examples 1.15 to 1.17, this is quite easy, because you can manually go though each element, one by one, and verify that it is in the other set.

The claims in part (2) of these examples are harder to check because each set in question contains infinitely many elements. Trying to manually consider each element of, say $[5,10]\cap[8,16]$, one by one, and check that it is in $[8,10]$, and vice-versa, would become thoroughly upsetting and ultimately fruitless. What is needed instead is an argument that applies simultaneously and equally well to **all** elements in question. Such an argument will not be supplied here, but it is important to be aware of its necessity.